Unfolding

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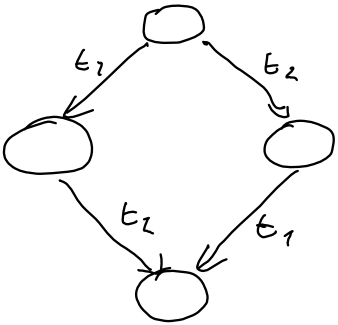
Partial-order Semantics

Introduced for Petri nets in 1980. G. Winskel

The intuition is that events (transitions) are allowed to happen concurrently, thus the firing “sequence” is only a partial order.



concurrency

Petri Nets Reminder

We limit ourselves to (1-)safe Petri nets with no sink transitions.

$\forall t \in T, {}^\bullet t \neq \emptyset \qquad \forall t \in T, t^\bullet \neq \emptyset$

Additionally, as we do not need to consider arc weights, we define the set of arcs as a function $W : P \cup T \times T \cup P \to \{0, 1\}$, whose value is 1 if the corresponding arc is present and 0 otherwise.

$W(p, t) = 1 \qquad p \in {}^\bullet t$

$M \xrightarrow{t} M' \qquad \forall p \in P \qquad M'(p) = M(p) - W(p, t) + W(t, p)$

Branching Processes – Intuition

Formalisation of partial-order semantics of Petri nets.

"Branching and acyclic" structure. Typically infinite.

The process of unfolding a Petri net results in possibly infinitely many different branching processes, depending on when is the process terminated. The, unique up to isomorphism, maximal branching process is called **the unfolding** of the Petri net.

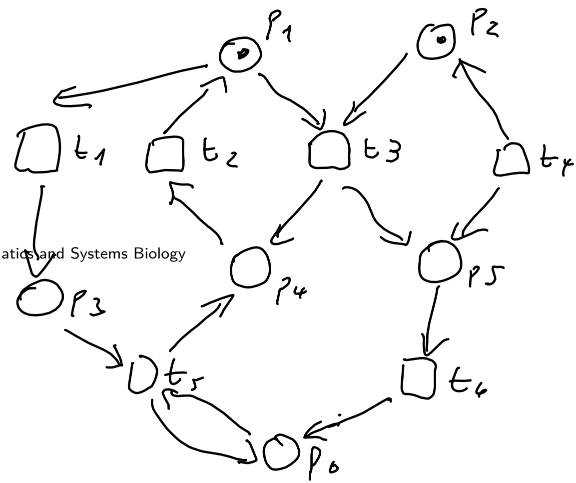
Causality, Conflict and Concurrency

Let $x, y \in P \cup T$ be two elements of a Petri net structure.

x and y are in causal relation, $x \leq y$, $\iff$ exists a path of arcs from x to y.

x and y are in conflict (relation), $x \# y$, $\iff$ exists a place $p \in P$ and two transitions $t_1 \neq t_2 \in T$ such that $p \in {}^\bullet t_1 \cap {}^\bullet t_2$ and there exist arc paths from t_1 to x and t_2 to y.

x and y are in concurrency relation (concurrent), $x \, co \, y$, $\iff$ $\neg(x \leq y) \wedge \neg(y \leq x) \wedge \neg(x \# y)$



Occurrence Nets

events

An occurrence net is a Petri net structure $O = (B, E, F)$ fulfilling the following properties:

conditions

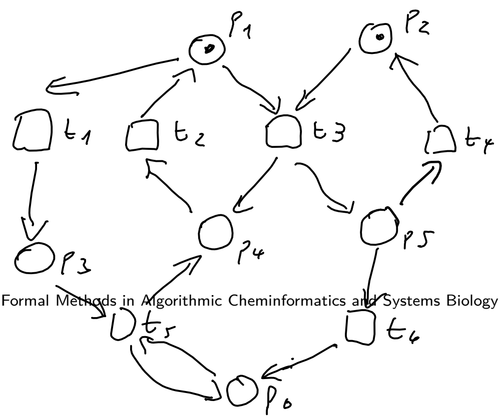
1. $\forall b \in B, |^{\bullet}b| \leq 1$;
2. The causal relation is a partial order (O is acyclic);
3. $\forall x \in B \cup E, \{y \in B \cup E \mid y < x\}$ is finite (O is finitely preceded);
4. The conflict relation is irreflexive;

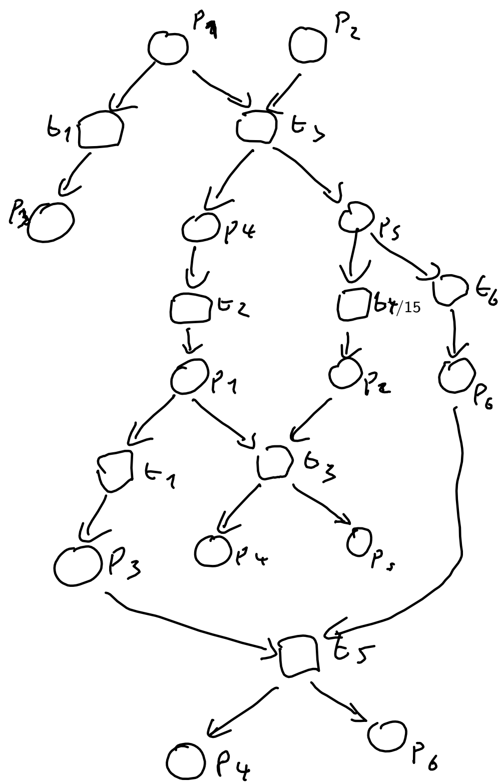
$$Min(O) = \{ b \in B \mid |^{\bullet}b| = 0 \}$$

Branching Processes – Formally

A branching process of a Petri net (P, T, W, M_0) is a labelled occurrence net $\beta = (O, \lambda) = (B, E, F, \lambda)$ where the labelling function $\lambda\colon B \cup E \to P \cup T$ satisfied the following properties:

1. $\lambda(B) \subseteq P$ and $\lambda(E) \subseteq T$;
2. $\forall e \in E$, $\lambda({}^\bullet e) = {}^\bullet\lambda(e)$ and $\lambda(e^\bullet) = \lambda(e)^\bullet$;
3. $\lambda(Min(O)) = M_0$;
4. $\forall e_1, e_2 \in E$, ${}^\bullet e_1 = {}^\bullet e_2 \wedge \lambda(e_1) = \lambda(e_2) \implies e_1 = e_2$;





Configurations and Cuts

Configuration of a branching process $\beta = (B, E, F, \lambda)$ is a set of events $C \subseteq E$ such that:

1. $e \in C \Longrightarrow \forall e' < e,\ e' \in C$;
2. $\forall e, e' \in C, \neg(e \# e')$;

$\approx$ "firing sequences"

Cut is a maximal set of pairwise concurrent conditions (co-set).

$\approx$ "Reachable markings"

$$Cut(C) = (Min(O) \cup C^{\bullet}) \setminus {}^{\bullet}C$$

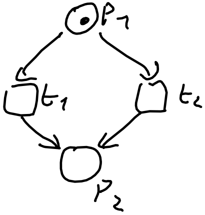
$$Mark(C) = \lambda(Cut(C))$$

Possible Extensions

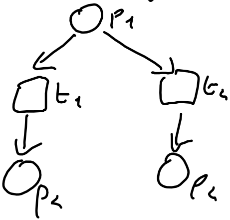
A possible extension of a branching process $\beta = (B, E, F, \lambda)$ of a Petri net (P, T, W, M_0) is a pair (t, X) where $t \in T$ is a transition and $X \subseteq B$ is a co-set of conditions such that $\lambda(X) = {}^\bullet t$ and $\forall e \in E$, $X = {}^\bullet e \Longrightarrow \lambda(e) \neq t$.

Complete Prefix

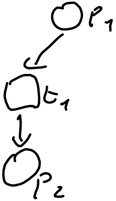
A branching process β of (N, M_0) is complete $\iff \forall M \in R(N, M_0)$, there exists a configuration C of β such that $Mark(C) = M$ and $\forall t \in T$, $M \models t \implies \exists e \in E \setminus C$ such that $\lambda(e) = t$ and $C \cup \{e\}$ is a configuration of β.



The unfolding



prefix



Cut-offs

McMillan (1995) introduced the concept of cut-off events for construction of complete finite prefixes.

A cut-off event e is such that no events causally dependent on e get added to the branching process.

C is a conf. of $\beta = (O, \lambda)$

$\Uparrow C = (O', \lambda')$ where O' is a subnet of O

identified by $\{x \in B \cup E \mid x \notin C \cup {}^\bullet C \wedge \forall y \in C, \neg(x \# y)\}$

"$\Uparrow C$ is everything that comes after C"

Branching process β of (N, M_0) and C a conf. of β

then $\Uparrow C$ is a branching process of $(N, Mark(C))$

C_1, C_2 such that $Mark(C_1) = Mark(C_2)$

$\Uparrow C_1 = \Uparrow C_2$ (up to isomorphism)

$C_1 \oplus E = C_2 \oplus I_1^2(E)$

$C_1 \oplus E \rightarrow$ new configuration

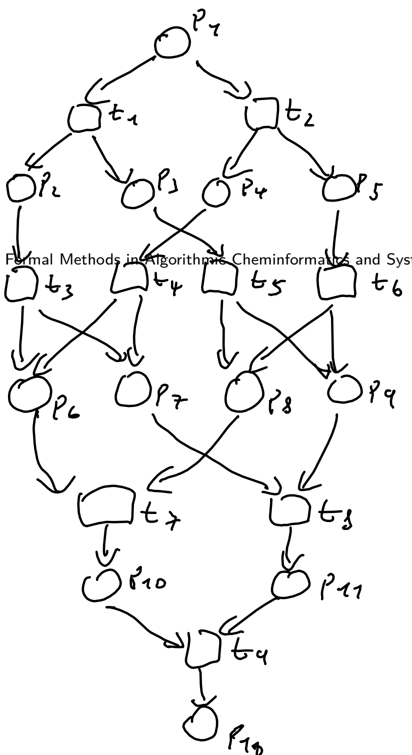
E is an extension of C_1 $E \cap C_1 = \emptyset$

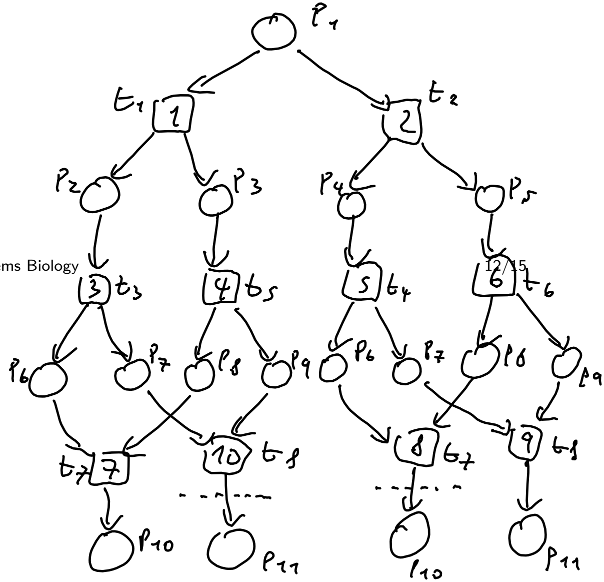
Local Configurations

For each event $e \in E$, the local configuration is the causal closure $[e] = \{e' \in E \mid e' \leq e\}$.

"The minimal set of events that must occur for e to be possible"

If we have $e, e' \in E$ s.t. $Mark([e]) = Mark([e'])$ then we could declare one of cut-off.





Adequate Order

A partial order $\prec$ on the configurations of a branching process β is **adequate** if it satisfied the following properties:

1. $\prec$ is well-founded;
2. $C_1 \subset C_2 \implies C_1 \prec C_2$;
3. $C_1 \prec C_2 \implies \forall$ finite extensions $C_1 \oplus E$, $C_1 \oplus E \prec C_2 \oplus I_1^2(E)$;

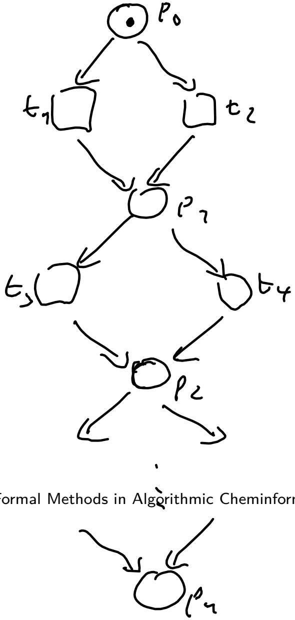
"finite extensions preserve $\prec$"

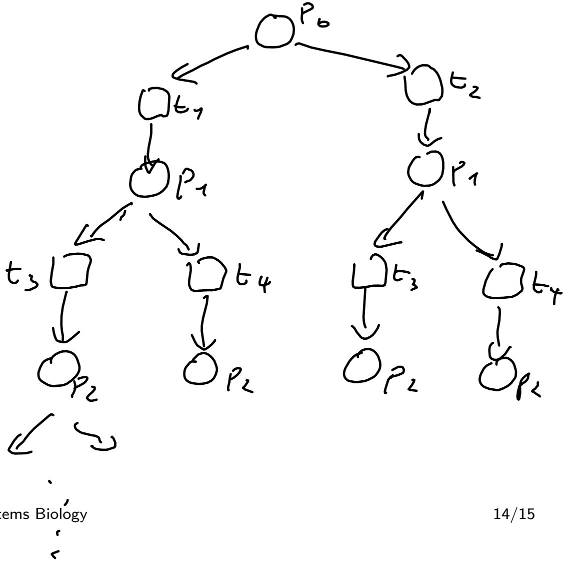
An event $e \in E$ is a cut-off $\iff \exists e' \in E$ such that $Mark([e']) = Mark([e])$ and $[e'] \prec [e]$.

$$[e] \prec [e'] \iff |[e]| < |[e']|$$

Size of the Complete Finite Prefix

Using the order on the sizes of the configurations as the adequate order (McMillan, 1995), the complete finite prefix can be up to exponentially larger than the reachability tree.





2^n copies of p_n

Total Adequate Order

$C_1 \prec C_2$ if one of the following holds:

1. $|C_1| < |C_2|$;
2. $|C_1| = |C_2|$ and $\varphi(C_1) \ll \varphi(C_2)$;
3. $\varphi(C_1) = \varphi(C_2)$ and $FC(C_1) \ll FC(C_2)$;

$\Rightarrow$ The number of non-cut-off events in the complete finite prefix is at most the number of reachable markings

Let $\ll \subseteq T \times T$ be a total order on the transitions

$\varphi(C)$ is the sequence of $\lambda(e)$ for all $e \in C$ ordered by $\ll$

$\varphi(C_1) \ll \varphi(C_2)$ is the lexicographic order (based on $\ll$)

$FC(C)$ is the Foata normal form

$FC(C) = C_1 \dots C_n \quad \forall e \in C_1, \forall e' \in C \quad \neg(e' < e)$

for $1 < i \leq n$, $\forall e \in C_i$, $\forall e' \in C$,

$e' < e \Rightarrow \exists j < i,\ e' \in C_j$

$FC(C_1) \ll FC(C_2) \iff \exists 1 \leq i \leq n_1$ such that

$C_{11} \dots C_{1n_1} \quad C_{21} \dots C_{2n_2}$

$\forall 1 \leq j < i,\ \varphi(C_{1j}) = \varphi(C_{2j})$

and $\varphi(C_{1i}) \ll \varphi(C_{2i})$