Stochastic Petri Nets

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Probability Theory Refresher

Set of events is exhaustive if they cover the entire sample space.
Events are mutually exclusive if they cannot co-occur.

Conditional probability $P(A|B) = \frac{P(AB)}{P(B)}$

Statistical independence $P(AB) = P(A) \cdot P(B)$

Theorem of total probability $P(B) = \sum_{i=1}^{n} P(A_i B)$

$$P(B) = \sum_{i=1}^{n} P(B|A_i) \cdot P(A_i)$$

Random variables and their moments

Geometric variable

$$p_X(k) = (1-p)^{k-1} p$$

Exponentially distributed variable (cont.)

$$\rightarrow f_X(x) = \lambda e^{-\lambda x} \qquad F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(x)\,dx = 1 - e^{-\lambda x}$$

probability density function cumulative distribution function

$$E(X) = \frac{1}{\lambda} = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx$$

variance (2nd central moment)

$$\sigma_X^2 = E(X^2) - (E(X))^2$$

Stochastic Process

A family of random variables $\{X(t)\}$ each with the same set of possible values. state set $S = \{1, 2, \dots\}$

Classification on state space:

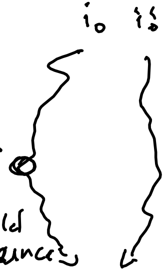
- Discrete; ⇝ stochastic chain
- Continuous;

Classification on time:

- Discrete; more illustrative
- Continuous; more used

Classification on the nature of the joint probability distribution function.

↳ Markov processes



Markov Chains

Markov property:

$$P(X(t) = j | X(t_n) = i_n, \ldots, X(t_0) = i_0) = P(X(t) = j | X(t_n) = i_n)$$

$t > t_n > \ldots > t_0$ $\quad$ $i_0, \ldots, i_n, j \in S$

there cannot be a side effect that would influence the probability in of successor states of i_n

A Markov chain is homogeneous iff:

$$P(X(t+s) = j | X(u+s) = i) = P(X(t) = j | X(u) = i)$$

$t \geq u$ $\quad$ $s \geq 0$

Discrete Time Markov Chains

$$P(X_{n+1}=i_{n+1} \mid X_n=i_n, X_{n-1}=i_{n-1}, \ldots, X_0=i_0) =$$

$$= \underbrace{P(X_{n+1}=i_{n+1} \mid X_n=i_n)}$$

One-step transition probabilities

$$p_{ij}(n,n+1) = P(X_{n+1}=j \mid X_n=i)$$

$$p_{ij} = p_{ij}(1) = P(X_{n+1}=j \mid X_n=i) \quad n \in \mathbb{N}_0$$

m-step transition probabilities

$$p_{ij}(m) = P(X_{n+m}=j \mid X_n=i)$$

$$\forall n \in \mathbb{N}_0 \, \forall i \in S \quad \sum_{j \in S} p_{ij}(m) = 1$$

Chapman-Kolmogorov equations for DTMC

$$P(X_{n+m}=j \mid X_n=i) = \sum_{k \in S} P(X_{n+m}=j \mid X_{n+r}=k, X_n=i) \cdot P(X_{n+r}=k \mid X_n=i)$$

$$m \geq r \geq 0$$

$$p_{ij}(m) = \sum_{k \in S} p_{kj}(m-r) \cdot p_{ik}(r)$$

$$p_{ij}(m) = \sum_{k \in S} p_{kj}(m-1) \cdot p_{ik}$$

$$P = [p_{ij}] \qquad P(m) = P(m-1) \cdot P$$

$$P(m) = P^m \qquad P^0 = I$$

Time-Based State Distribution

Probability of being in a state j at time (step) m. $\pi_j(m) = P(X_m = j)$

$\pi_j(0) \Rightarrow$ initial distribution

$$\sum_{i \in S} \pi_i(0) \cdot p_{ij}(m) = \sum_{i \in S} P(X_0 = i) \cdot P(X_m = j \mid X_0 = i) =$$

$$= P(X_m = j) = \pi_j(m)$$

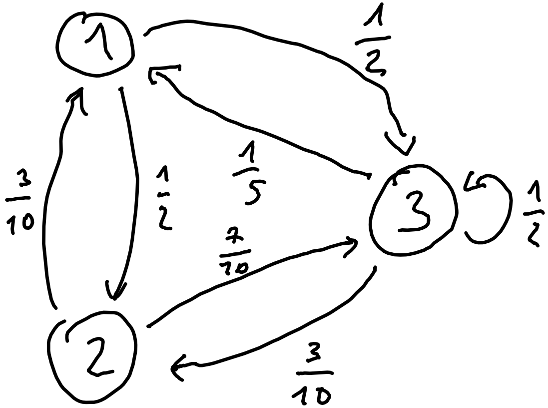
(Theorem of total prob.)

$$\Pi(m) = \Pi(0) P^m$$

$$\pi_j = \lim_{m \to \infty} P(X_m = j)$$

Does this limit exist?

If so, does it depend on $\Pi(0)$?



$$P = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.3 & 0 & 0.7 \\ 0.2 & 0.3 & 0.5 \end{pmatrix}$$

$$\pi(0) = (1, 0, 0)$$

$$\pi(1) = (1, 0, 0) \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.3 & 0 & 0.7 \\ 0.2 & 0.3 & 0.5 \end{pmatrix} = (0, 0.5, 0.5)$$

$$\pi(2) = (1, 0, 0) \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.3 & 0 & 0.7 \\ 0.2 & 0.3 & 0.3 \end{pmatrix}^2$$

$$= (1, 0, 0) \begin{pmatrix} 0.25 & 0.15 & 0.6 \\ 0.14 & 0.36 & 0.5 \\ 0.19 & 0.25 & 0.56 \end{pmatrix} = (0.25 \quad 0.15 \quad 0.6)$$

Structural Classifications of Markov Chains

Two states **communicate** iff there exist directed paths between them.

$$i \rightleftarrows j \iff \exists\, n, n' \in \mathbb{N} \text{ s.t. } p_{ij}(n) > 0 \wedge p_{ji}(n') > 0$$

The class of a state i is the set of all states which communicate with i.

$$C(i) = \{ j \in S \mid i \rightleftarrows j \}$$

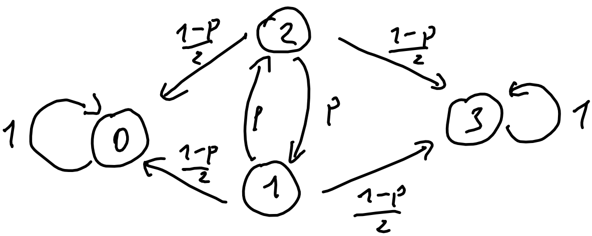
A Markov chain is **irreducible** if it consists of a single class.

A class C is **transient** if there exists a non-zero one-step transition probability leading out of C.

$$\exists\, i \in C, \exists\, j \notin C \text{ s.t. } p_{ij} > 0$$

A class C is **ergodic** if any path starting in C, remains in C.

$$\forall i \in C, \sum_{j \in C} p_{ij} = 1$$



$C(0) = \{0\}$

$C(3) = \{3\}$

$C(1) = C(2) = \{1, 2\}$

Recurrence Classification of Markov Chains

Let $f_j(m)$ be the probability of leaving a state j and first returning in m steps. Then the probability of ever returning to j is:

$$f_j = \sum_{m=1}^{\infty} f_j(m)$$

The state j is:

- Transient if $f_j < 1$;
- Recurrent if $f_j = 1$;
- Periodic if the return is only possible at steps $\nu, 2\nu, 3\nu, \ldots$ where ν is the largest such integer;

Mean recurrence time

$$M_j = \sum_{m=1}^{\infty} m \cdot f_j(m)$$

Additionally, a recurrent state can be:

- null recurrent if $M_j = \infty$;
- positive (non-null) recurrent if $M_j < \infty$;

Steady State Distribution

In an irreducible DTMC, all states are:

- transient;
- null recurrent;
- positive recurrent;

And if periodic, then all states have the same period.

$$\pi_j = \lim_{n \to \infty} P(X_n = j)$$

In an irreducible, aperiodic (and homogeneous) MC, the limit probabilities exist, are independent of the initial distribution and:

- all states are, either, transient or null recurrent and $\forall j \in S, \pi_j = 0$ (there exists no steady state distribution);
- all states are positive recurrent and $\forall j \in S, \pi_j = \frac{1}{M_j}$;

$$\sum_{i \in S} \pi_i = 1$$

$$\forall j \in S \quad \sum_{i \in S} \pi_i \cdot p_{ij} = \pi_j \approx \Pi \cdot P = \Pi$$

$J_i(n)$ .. Time spent in state i over an interval of length n

$$E(J_i) = \pi_i \cdot n$$

mean time spent in state i in-between two successive visits to state j

$$v_{ij} = \frac{\pi_i}{\pi_j}$$

Absorbing Chains

Order the states so that the absorbing states are first, followed by the transient ones.

$$P = \begin{pmatrix} I & 0 \\ R & Q \end{pmatrix} \qquad P^n = \begin{pmatrix} I & 0 \\ R \cdot \sum_{i=1}^{n} Q^{i-1} & Q^n \end{pmatrix}$$

$$R\left(I + Q + Q^2 + \ldots + Q^{n-1}\right)$$

$$n \to \infty \Rightarrow Q^n \to 0$$

$$\sum_{i=1}^{n} Q^{i-1} \to (1 - Q)^{-1} = N$$

fundamental matrix of the MC

$\delta_{ii} = 1$

$\delta_{ij} = 0 \quad i \neq j$

Time before absorption

Starting from state i, let v_{ij} be the number of visits to state j before an absorbing state is reached.

Kronecker delta function

$$E(v_{ij}) = n_{ij} \qquad E(v_{ij}) = \delta_{ij} + \sum_{k \in S_t} q_{ik} \cdot E(v_{kj})$$

$$M = [E(v_{ij})]$$

$$M = I + Q \cdot M$$

$$M - QM = I$$

$$(I - Q)M = I$$

$$M = (I - Q)^{-1} = N$$

$$v_i = \sum_{j \in S_t} v_{ij}$$

$$\tau_i = E(v_i) = \sum_{j \in S_t} E(v_{ij}) = \sum_{j \in S_t} n_{ij} \qquad \vec{\tau} = [\tau_i]$$

$$\vec{\tau} = N \cdot e^T \qquad e = [1]$$

$$\tau = \Pi_t(0) \cdot \vec{\tau} = \Pi_t(0) N \cdot e^T$$

mean time before absorption

$$\vec{\sigma}^2 = (2N - I)\vec{\tau} - \vec{\tau}_2 \qquad \vec{\tau}_2 = [E(v_i^2)]$$

Continuous-time Markov Chains

$$P(X(t)=j \mid X(t_n)=i_n, X(t_{n-1})=i_{n-1}, \dots, X(t_0)=i_0) =$$

$$= P(X(t)=j \mid X(t_n)=i_n)$$

$$t > t_n > \dots > t_0$$

$$p_{ij}(s,t) = P(X(t)=j \mid X(s)=i)$$

Chapman-Kolmogorov equations for CTMC

$$p_{ij}(t) = P(X(s+t)=j \mid X(s)=i)$$

$$p_{ij}(t) = \sum_{k\in S} p_{ik}(u)\, p_{kj}(t-u) \qquad p_{ij}(0) = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$$

$$0 \le u \le t$$

$$H(t) = [p_{ij}(t)]$$

$$H(t) = H(u) \cdot H(t-u)$$

$$H(0) = I$$

$$p_{ij}(t+\Delta t) = \sum_{k\in S} p_{ik}(t) \cdot p_{kj}(\Delta t) \qquad / - p_{ij}(t)$$

$$p_{ij}(t) = \sum_{k\in S} p_{ik}(t) \cdot p_{kj}(0)$$

$$p_{ij}(t+\Delta t) - p_{ij}(t) = \sum_{k\in S} p_{ik}(t) \cdot p_{kj}(\Delta t) - p_{ij}(t)$$

$$p_{ij}(t+\Delta t) - p_{ij}(t) = \sum_{k\in S} p_{ik}(t)\left(p_{kj}(\Delta t) - p_{kj}(0)\right) \qquad / \Delta t$$

$$\frac{p_{ij}(t+\Delta t) - p_{ij}(t)}{\Delta t} = \frac{\sum_{k\in S} p_{ik}(t)\left(p_{kj}(\Delta t) - p_{kj}(0)\right)}{\Delta t}$$

$$\frac{\delta H(t)}{\delta t} = H(t) \cdot Q$$

$$\frac{\delta H(t)}{\delta t} = H(t) \cdot Q$$

$$Q = \lim_{\Delta t \to 0} \frac{H(\Delta t) - I}{\Delta t}$$

$$q_{ij} = \lim_{\Delta t \to 0} \frac{p_{ij}(\Delta t)}{\Delta t} \qquad i \neq j$$

$$q_{ii} = \lim_{\Delta t \to 0} \frac{p_{ii}(\Delta t) - 1}{\Delta t}$$

$$\forall i \in S, \sum_{j \in S} q_{ij} = 0$$

Time-Based State Distribution (CTMC)

$$\pi_i(t) = P(X(t) = i)$$

$$\Pi(t) = \Pi(0) . H(t)$$

$$\frac{\delta \Pi(t)}{\delta t} = \Pi(0) \frac{\delta H(t)}{\delta t} =$$

$$= \Pi(0) . H(t) . Q$$

$$= \Pi(t) . Q$$

Steady State Distribution (CTMC)

The limit probabilities and the steady state distribution exist if the CTMC is irreducible, positive recurrent (and homogeneous).

$$\pi_j = \lim_{t \to \infty} \pi_j(t) = \lim_{t \to \infty} P(X(t) = j)$$

$$\sum_j \pi_j = 1$$

$$-q_{jj} \pi_j + \sum_{k \neq j} q_{kj} \pi_k = 0$$

SS

$$\pi Q = 0$$

$$\left(\overset{\text{discrete case}}{\pi \cdot P = \pi} \right)$$

Embedded DTMC

A CTMC might reimagined as a DTMC in which the transitions don't happen at equal "unit" intervals, but rather ones sampled from an exponential distribution.

Before transitioning from i to j we sojourn in i for time τ_{ij}, an exponentially distributed random variable

$$P(\tau_{ij} \leq t) = 1 - e^{-q_{ij} t} \qquad P(\tau_{ij} > t) = e^{-q_{ij} t}$$

τ_{min} = Time until any $k \in S, k \neq j$ fires

$$P(\tau_{min} \leq t) = P\Big(\bigcup_{k \neq i,j} (\tau_{ik} \leq t)\Big) =$$

$$= 1 - P\Big(\bigcap_{k \neq i,j} (\tau_{ik} > t)\Big)$$

$$= 1 - e^{-\sum_{k \neq i,j} q_{ik} \cdot t}$$

$$-q_{ii} = \sum_{k \neq i} q_{ik} \qquad \sum_{k \neq i,j} q_{ik} = -q_{ii} - q_{ij}$$

$$p_{ij} = \int_0^\infty f_{\tau_{min}}(t) \cdot P(\tau_{ij} \leq t)\, dt =$$

$$= -\int_0^\infty (q_{ii} + q_{ij}) \cdot e^{(q_{ii} + q_{ij}) t} \cdot \left(1 - e^{-q_{ij} t}\right) dt$$

$$= \frac{q_{ij}}{-q_{ii}}$$

τ_i = the sojourn time in state i, irrespective of the next transition.

$$\phi_j = \lim_{t \to \infty} P(X(t) = j)$$

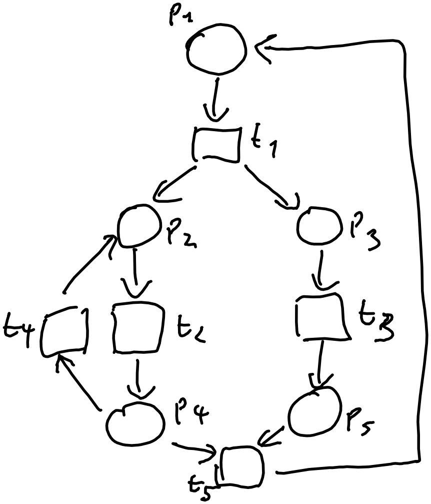
$$\phi_j = \frac{\pi_j \, E(\tau_j)}{\sum_{i=S} \pi_i \, E(\tau_i)} =$$

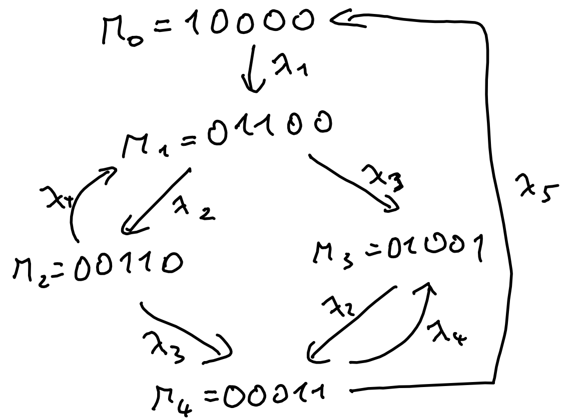
where π is the steady state distribution of the embedded DTMC

$$= \frac{\pi_j \cdot \frac{1}{-q_{jj}}}{\sum_{i=S} \pi_i \frac{1}{-q_{ii}}}$$

Stochastic Petri Nets

A stochastic Petri net is composed of a Petri net (N, M_0) and a collection $\Lambda = (\lambda_1, \ldots, \lambda_m)$ of, possibly marking dependent, transition rates of each transition $t_i \in |T|$.





In M_1, what is the probability that t_2 fires?

$P(X_2 < X_3)$ where X_2, X_3 are the random variables representing time until t_2, t_3 fires in M_1

$$= \frac{\lambda_2}{\lambda_2 + \lambda_3}$$

$$P(\underbrace{\min(X_2, X_3)}_{T_{M_1}} \le t) = P(X_2 \le t \cup X_3 \le t)$$
$$= 1 - P(X_2 > t \cap X_3 > t)$$
$$= 1 - P(X_2 > t) \cdot P(X_3 > t)$$
$$= 1 - e^{-\lambda_2 t} \cdot e^{-\lambda_3 t}$$
$$= 1 - e^{-(\lambda_2 + \lambda_3) t}$$

$$E(T_{M_1}) = \frac{1}{\lambda_2 + \lambda_3}$$