Petri Nets

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Brief History

Introduced in 1962 by Carl Adam Petri.

Introduced as a model of concurrency in the setting of parallel and distributed computing.

Biological and chemical systems are very often inherently concurrent.

Definition - syntax

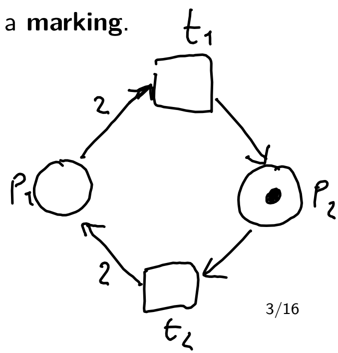
A directed bipartite graph with round vertices (**places**) and square vertices (**transitions**).

The edges (**arcs**) are labelled by a **weight** function.

The places may contain **tokens**, representing available resources.

The distribution of tokens over all places constitutes a **marking**.

$$\overbrace{(P, T, F, W}^{\text{Petri net structure } N}, M_0)$$



P ... finite set of places

T ... finite set of transitions

$$F \subseteq P \times T \cup T \times P$$

$$W : T \to \mathbb{N}$$

$$M_0 : P \to \mathbb{N}_0$$

$$M_0 = (2, 0)$$

$$(N, M_0)$$

$\forall x \in P \cup T \quad {}^\bullet x = \{ y \in P \cup T \mid (y, x) \in F \}$

"preset"

$\forall x \in P \cup T \quad x^\bullet = \{ y \in P \cup T \mid (x, y) \in F \}$

"poset"

Definition -semantics

A transition is **enabled** in a marking, if each of the input places has at least as many tokens as indicated by the arc weights.

An enabled transition may **fire**, removing tokens from the source places and producing tokens in the target places, according to the weight function.

$t \in T$ is enabled in $M \iff \forall p \in {}^\bullet t\,,\ M(p) \geq W(p,t)$

$M \vdash t$ (non-standard)

Firing $t \in T$ in M leads to M' $\qquad M \xrightarrow{t} M'$

$$\forall p \in P \quad M'(p) = \begin{cases} M(p) - W(p,t) + W(t,p) & \text{if } p \in {}^\bullet t \cap t^\bullet \\ M(p) + W(t,p) & \text{if } p \in t^\bullet \setminus {}^\bullet t \\ M(p) - W(p,t) & \text{if } p \in {}^\bullet t \setminus t^\bullet \\ M(p) & \text{if } p \notin {}^\bullet t \cup t^\bullet \end{cases}$$

${}^\bullet t = \emptyset$ "source"

$t^\bullet = \emptyset$ "sink"

Reachability

A sequence of transitions which can be fired in order is called a **firing sequence**.

A marking is **reachable**, if there exists a firing sequence producing it.

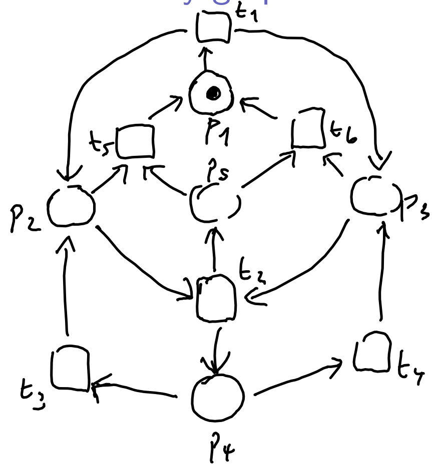
$\sigma = (t_1, \ldots, t_k)$

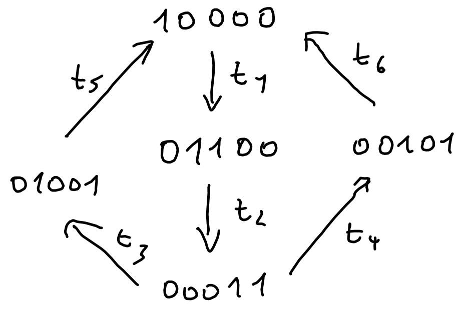
$M \xrightarrow{t_1} M_1 \xrightarrow{t_2} \ldots \xrightarrow{t_k} M_k \qquad M \xrightarrow{\sigma} M_k$

Set of reachable markings $R(N, M_0) =$

$= \{ M \mid \exists \sigma, M_0 \rightarrow M \}$

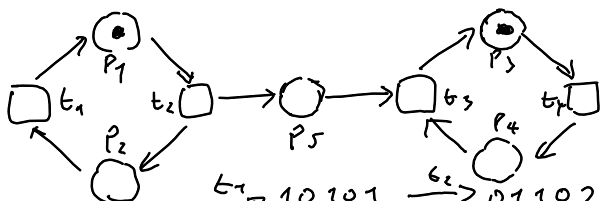
Reachability graph





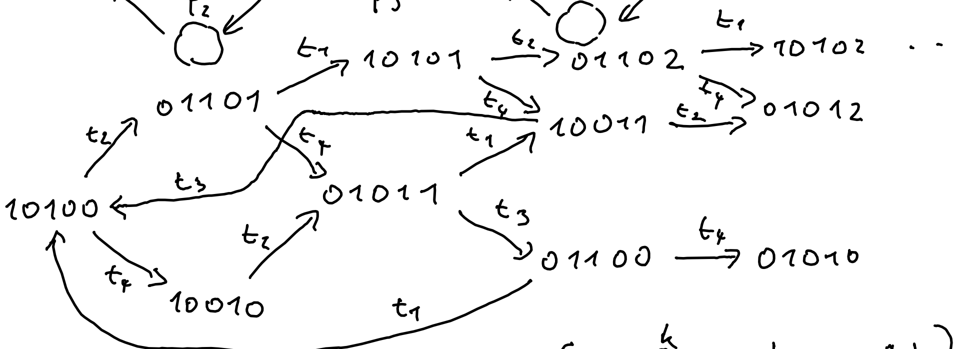
$M_0: (1,0,0,0,0)$

safe (1-bounded)



$M_0: (1,0,1,0,0)$

unbounded



$$\{r + \sum_{i=1}^{k} p_k \cdot \lambda_k \mid \lambda_k \in \mathbb{N}_0\}$$

2012, Leroux

Coverability graph

A M is coverable $\iff \exists M' \in R(N, M_0)$ such that $M' \geq M$

Monotonicity property: $M_1 \leq M_2 \Rightarrow R(N,M_1) \subseteq R(N,M_2)$

ω-markings:

$M: P \to \mathbb{N}_0 \cup \{\omega\}$ $\quad \forall x \in \mathbb{N}_0: x < \omega$

$x + \omega = \omega$

$\omega - x = \omega$

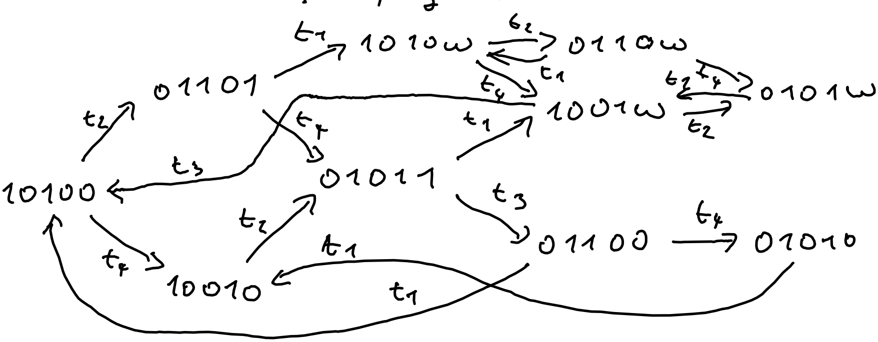
$M_2 \in R(N,M_0)$ $\quad \exists M_1 \in R(N,M_0) \wedge \exists \sigma_2$ such that

$$M_0 \xrightarrow{\sigma_1} M_1 \xrightarrow{\sigma_2} M_2 \quad \wedge \quad M_1 \leq M_2$$

then we introduce an ω-marking M_2' defined as

$$\forall p \in P \quad M_2'(p) = \begin{cases} \omega & \text{if } M_2(p) > M_1(p) \\ M_2(p) & \text{otherwise} \end{cases}$$

This follows from σ_2 being firable from M_2 as well, thus "pumping" tokens into places $M_2(p) > M_1(p)$



Boundedness

A Petri net is **k-bounded** if in every reachable marking, the number of tokens in every place is at most k.

A Petri net is **bounded** if it k-bounded for some $k \in \mathbb{N}$.

1-bounded Petri nets are also known as (1-)**safe** Petri nets.

For $k \in \mathbb{N}$

$p \in P$ is k-bounded $\iff \forall M \in R(N, M_0), M(p) \leq k$

(N, M_0) is k-bounded $\iff \forall p \in P$ p is k-bounded

safe Petri nets are qualitative, as opposed to quantitative.

Deadlocks

Deadlock is a marking that enables no transitions.

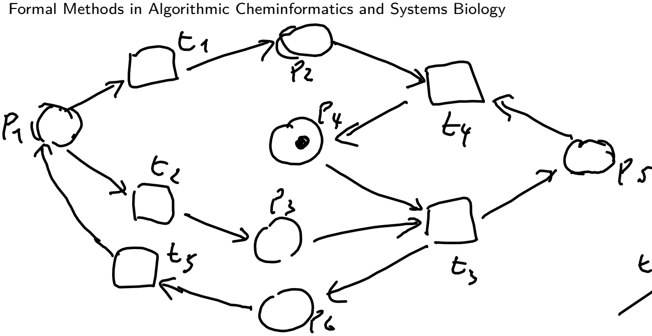
A Petri net is **deadlock-free** if no reachable marking is a deadlock.

$M \in R(N, M_0)$ is a deadlock $\iff \forall t \in T,\ M \not\vdash t$

(N, M_0) is deadlock-free $\iff \forall M \in R(N, M_0)$

M is not a deadlock

In bio/chem. a deadlock is "steady state"



$M_0 : (1, 0, 0, 1, 0, 0)$

$100100 \xrightarrow{t_1} \underline{010100}$

$100100 \xrightarrow{t_2} 001100 \xrightarrow{t_3} 000011 \xrightarrow{t_5} 100010$

$100010 \xrightarrow{t_1} 010010 \xrightarrow{t_4} \underline{000100}$

$100010 \xrightarrow{t_2} \underline{001010}$

Liveness

A Petri net is **live** if no firing sequence permanently disables any transition.

$\forall M \in R(N, M_0), \forall t \in T, \exists M' \in R(N, M)$ such that $M' \vdash t$

A transition t is:

L0-live if t can never be fired (dead);

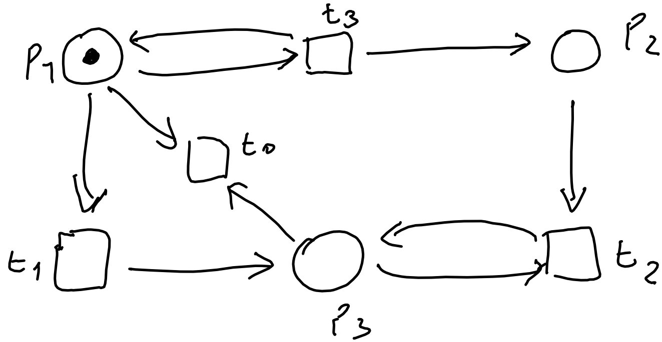
L1-live if t can be fired at least once (potentially fireable); $\exists M \in R(N, M_0)$ such that $M \vdash t$

L2-live if given $k \in \mathbb{N}$, t can be fired at least k times; $\exists \sigma, M_0 \xrightarrow{\sigma} M'$, t appears k times in σ

L3-live if t appears infinitely often in some firing sequence;

L4-live if t is L1-live for every reachable marking (live).

$\exists \sigma$ $M_0 \xrightarrow{\sigma} M$ such that t appears in σ and σ is firable in M



$M_0 : (1, 0, 0)$

Complexity

Petri net problems are EXPSPACE-hard.

Coverability and Boundedness are EXPSPACE-complete.

Reachability, Deadlock-freedom and liveness are non-elementary.

Safe Petri net problems are PSPACE-complete.

$exp(k) = \underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2^n}}}}}}_{k}$ $\qquad$ EXPSPACE $\sim \bigcup_{k \in \mathbb{N}} exp(k)$

k depends on the input => non-elementary

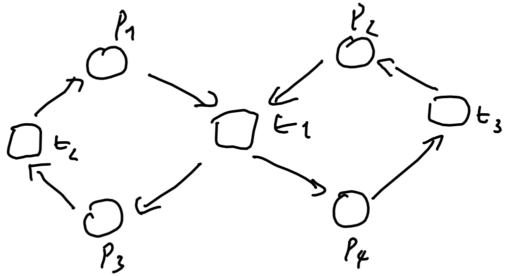
Static Analysis

A Petri net (structure) can be represented by an **incidence matrix** recording the tokens consumed and produced by each transition.

The results of a firing sequence can be obtained as matrix multiplication.

Necessary condition for reachability.

$$
\begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{array}
\begin{array}{c} \begin{array}{ccc} t_1 & t_2 & t_3 \end{array} \\ \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \end{array}
$$



$$
\begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}
$$

$$M_0 + A \cdot u = M$$

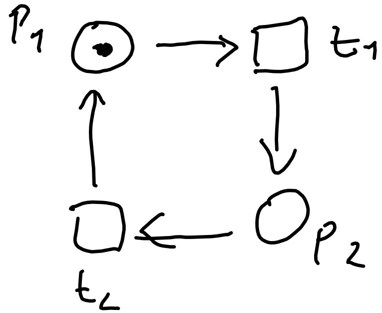
M is only reachable if such an u exists

↱ necessary

If such an u does not exist: M is not reachable

if such an u exists: M might still not be reachable.

↳ but not sufficient



$$\begin{array}{ccccc} M_0 & & A & \cdot\, u & & M \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} & + & \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} & \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} & = & \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{array}$$

$$1 + (-u_1 + u_2) = 1$$

$$0 + (u_1 - u_2) = 1$$

$$u_2 - u_1 = 0 \Rightarrow u_1 = u_2$$

$$u_1 - u_2 = 1 \Rightarrow \text{No solution}$$

Given the u, finding the corresponding firing sequence σ : Realisability

Place invariants

A **P-invariant** defines a linear combination of the numbers of tokens in places which is invariant.

$$x \cdot A = 0$$

$$(x_1, x_2, x_3, x_4) \cdot \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} = 0$$

$x_3 + x_4 - x_1 - x_2 = 0$

$x_1 - x_3 = 0 \Rightarrow x_1 = x_3$

$x_2 - x_4 = 0 \Rightarrow x_2 = x_4$

$(x, y, x, y) \quad x, y \in \mathbb{N}_0$

Minimal P-invariants

$(1, 0, 1, 0)$

$(0, 1, 0, 1)$

$M \in R(N, M_0)$ Let u such that $M = M_0 + A \cdot u$

$$\underbrace{x \cdot M = x \cdot M_0}$$

$$x \cdot M = x \cdot (M_0 + A \cdot u) = x \cdot M_0 + \underbrace{x \cdot A}_{0} \cdot u = x \cdot M_0$$

$x \cdot M_0 > 0$

$\shortparallel$

$x \cdot M \geq x_i \cdot M_i$

Transition invariants

A **T-invariant** defines a firing sequence that restores the original marking.

$$A \cdot y = 0$$

$$\begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = 0$$

Minimal invariant:

$(1, 1, 1)$

$$y_2 - y_1 = 0 \qquad y_1 - y_3 = 0$$

$$y_3 - y_1 = 0$$

$$\Rightarrow y_1 = y_2 = y_3$$

$$y_1 - y_2 = 0$$

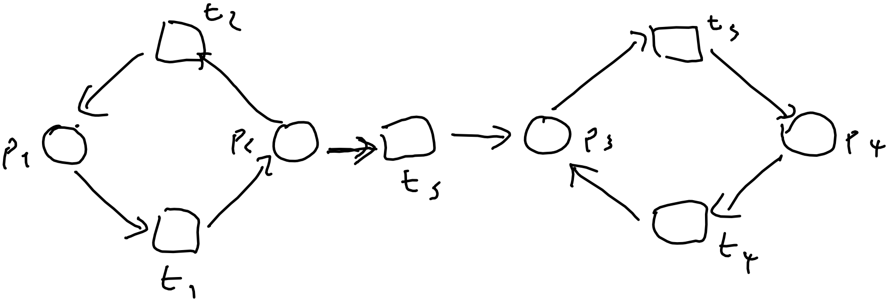
Siphons and Traps

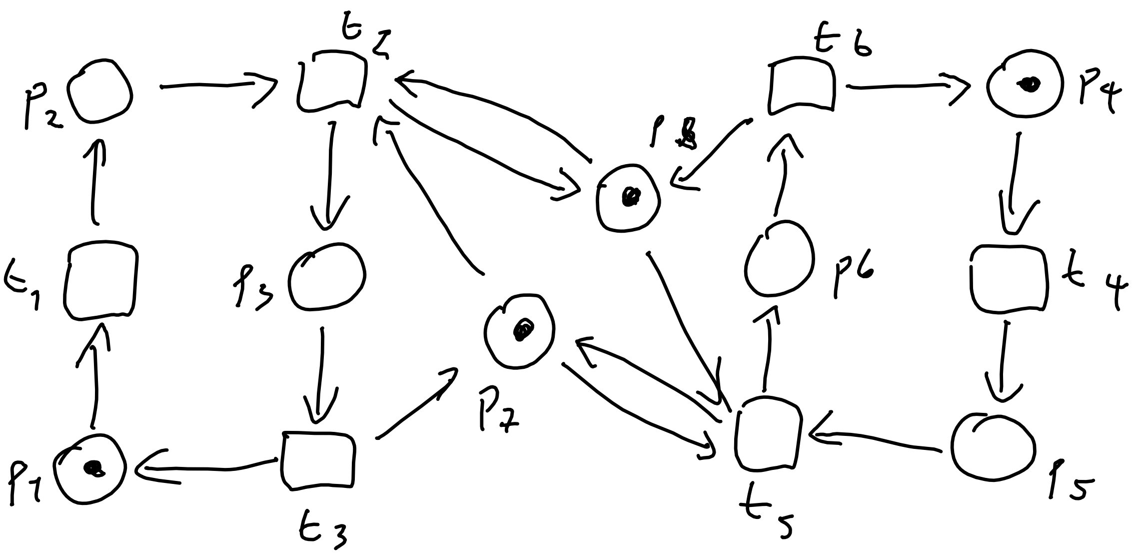
A **siphon** is a set of places that can only gain tokens if it also loses tokens.

$S \subseteq P$ is a siphon iff ${}^\bullet S \subseteq S^\bullet$

A **trap** is a set of places that can only lose tokens if it also gains tokens.

$S \subseteq P$ is a trap iff $S^\bullet \subseteq {}^\bullet S$





Goal: $M_3 + M_6 \le 1$

$M_1 + M_2 + M_3 = 1$

$M_4 + M_5 + M_6 = 1$

$M_3 + M_7 = 1$

$M_6 + M_8 = 1$

$S = \{p_7, p_8\}$ is a trap

$M_7 + M_8 \ge 1$

$M_3 + M_6 + M_7 + M_8 = 2$

$M_3 + M_6 - 2 = -M_7 - M_8$

$M_3 + M_6 - 2 \le -1$

$M_3 + M_6 \le 1$

Extensions

- Capacities; $K : P \to \mathbb{N}$
- Inhibitor arcs; $I \subseteq T \times P$

Break the monotonicity: $M_1 \le M_2 \Rightarrow R(N, M_1) \le R(N, M_2)$

- Read arcs; $R \subseteq T \times P$
- Timed Petri nets; $\tau : T \to \mathbb{N}$
- Stochastic Petri nets;
- Continuous Petri nets; $\rightsquigarrow$ ODEs
- Hybrid Petri nets;
- Fuzzy Petri nets;
- Coloured Petri nets.

Tokens can have different colours