Transitions Systems

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Transition Graphs

The semantics of a Boolean network f of dimension n are given by the transition system $\left(S = \mathbb{B}^n, \underset{f}{\longrightarrow}\right)$.

The transition system gives rise to a directed graph (V, E) where $V = S = \mathbb{B}^n$ and $E \subseteq V \times V = \underset{f}{\longrightarrow}$.

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$f_1(\mathbf{x}) = \mathbf{x}_3 \wedge (\neg \mathbf{x}_1 \vee \neg \mathbf{x}_2)$, $f_2(\mathbf{x}) = \mathbf{x}_1 \wedge \mathbf{x}_3$, $f_3(\mathbf{x}) = \mathbf{x}_1 \vee \mathbf{x}_2 \vee \mathbf{x}_3$

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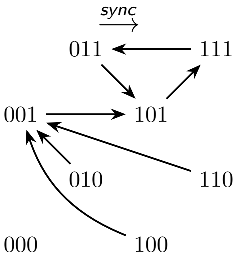

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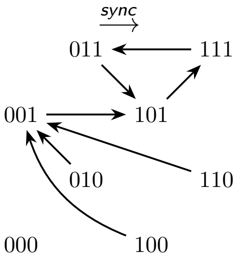

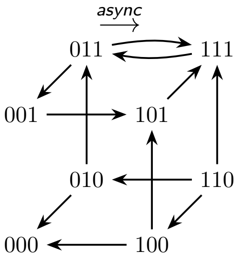

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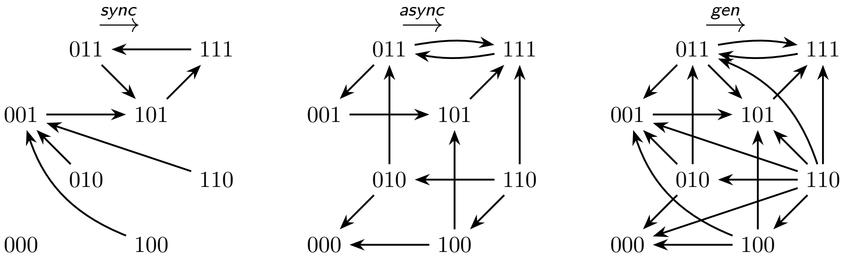

Reachability

A **trace** $\sigma = (\mathbf{x}^0, \mathbf{x}^1, \mathbf{x}^2, \dots)$ is an (infinite) sequence of configurations such that for all $k > 0$, $\mathbf{x}^{k-1} \to \mathbf{x}^k$.

We often write $\sigma = \mathbf{x}^0 \to \mathbf{x}^1 \to \mathbf{x}^2 \to \dots$.

Each trace represents a possible behaviour of the model, starting from the **initial state** $\mathbf{x}^0$.

A configuration $\mathbf{y}$ is **reachable** from $\mathbf{x}$ if and only if there exists a trace $\sigma = (\mathbf{x} = \mathbf{x}^0, \mathbf{x}^1, \mathbf{x}^2, \dots)$ and an integer $k \geq 0$ such that $\mathbf{x}^k = \mathbf{y}$.

$\forall x \in \mathbb{B}^n$ $\sigma = (x)$ witnesses reachability of x from x

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Alternatively, let $\to^* \subseteq \mathbb{B}^n \times \mathbb{B}^n$ be the reflexive and transitive closure of the semantics relation $\to$.

Then $\mathbf{y}$ is reachable from $\mathbf{x}$ if and only if $\mathbf{x} \to^* \mathbf{y}$.

reflexive: $\forall x \in \mathbb{B}^n : x \to^* x$

transitive $\forall x, y, z \in \mathbb{B}^n : x \to^* y \wedge y \to^* z \Rightarrow x \to^* z$

Trap Sets

A trap set is a non-empty set of configurations $\emptyset \neq T \subseteq \mathbb{B}^n$ which satisfies the following property:

$$\forall \mathbf{x} \in T, \forall \mathbf{y} \in \mathbb{B}^n, \mathbf{x} \rightarrow^* \mathbf{y} \Rightarrow \mathbf{y} \in T$$

Let $\mathcal{T}$ denote the set of all trap sets of a given BN.

Let $T, T' \in \mathcal{T}$ be two trap sets. We say T is smaller than T' if and only if it is included in T', $T \subseteq T'$.
For each configuration $\mathbf{x} \in \mathbb{B}^n$, let $[\mathbf{x}]_{\mathcal{T}} \triangleq \{\mathbf{y} \in \mathbb{B}^n \mid \mathbf{x} \rightarrow^* \mathbf{y}\}$ denote the smallest trap set containing $\mathbf{x}$.
A trap set T is minimal, if $\forall T' \in \mathcal{T}$, $T \not\supset T'$.

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Trap set properties

Closed by non-empty intersection: $T, T' \in \mathcal{T}$ such that $T \cap T' \neq \emptyset$, then $T \cap T' \in \mathcal{T}$.
Minimal trap sets are "strongly connected": $T \in \mathcal{T}$ minimal, then $\forall \mathbf{x}, \mathbf{y} \in T, \mathbf{x} \rightarrow^* \mathbf{y}$.

$T, T' \in \mathcal{T} \quad T \cap T' \neq \emptyset$ then $T \cap T' \in \mathcal{T}$

proof:

$\exists x \in T \cap T'$ arbitrary

$\forall y \in \mathbb{B}^n$ such that $x \rightarrow^* y$

$$\left.\begin{array}{l} T \in \mathcal{T} \land x \in T \Rightarrow y \in T \\ T' \in \mathcal{T} \land x \in T' \Rightarrow y \in T' \end{array}\right\} y \in T \cap T' \Rightarrow T \cap T' \in \mathcal{T}$$

$T \in \mathcal{T}$ is minimal, then $\forall x, y \in T, x \rightarrow^* y$

proof (by contradiction):

Let's assume $x \not\rightarrow^* y$

Then $T' = \{z \in \mathbb{B}^n \mid x \rightarrow^* z\}$ we know $T' \subset T$

$y \notin T'$

We want to show that $T' \in \mathcal{T}$, contradicting minimality of T.

Let $z \in T'$ arbitrary

$\forall z' \in \mathbb{B}^n$ such that $z \rightarrow^* z'$ we have $x \rightarrow^* z \rightarrow^* z' \Rightarrow x \rightarrow^* z'$

and thus $z' \in T' \Rightarrow T' \in \mathcal{T}$

Attractors

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1. $\forall \mathbf{x} \in A, \forall \mathbf{y} \in \mathbb{B}^n, \mathbf{x} \rightarrow^* \mathbf{y} \Rightarrow \mathbf{y} \in A$ (is a trap set);
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Types of attractors:

- **Fixed points** (singletons), $\forall \mathbf{x}, \mathbf{y} \in A, \mathbf{x} = \mathbf{y}$;
- **Cyclic attractors** (or limit cycles), consisting of at least two distinct configurations, $\exists \mathbf{x}, \mathbf{y} \in A, \mathbf{x} \neq \mathbf{y}$;

A configuration which is not part of any attractor is called **transient**.

Attractors – Example

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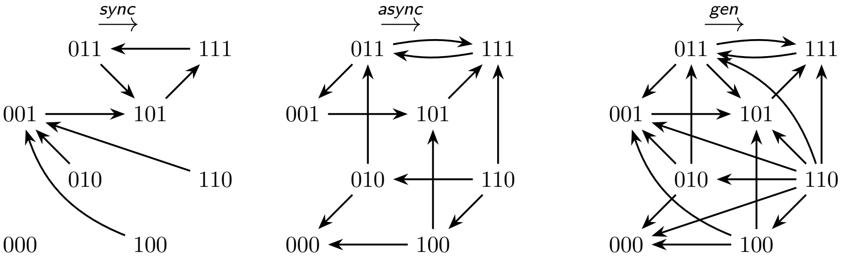

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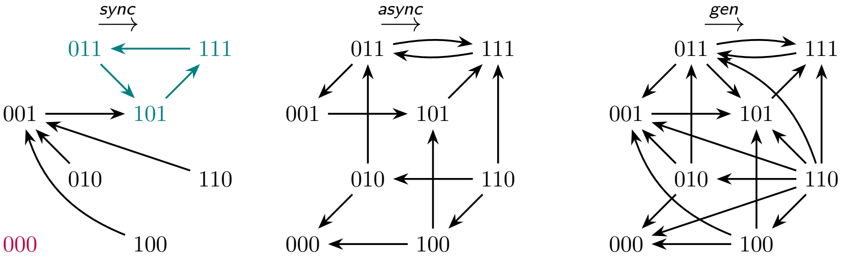

$A_1 = \{101, 011, 111\}$

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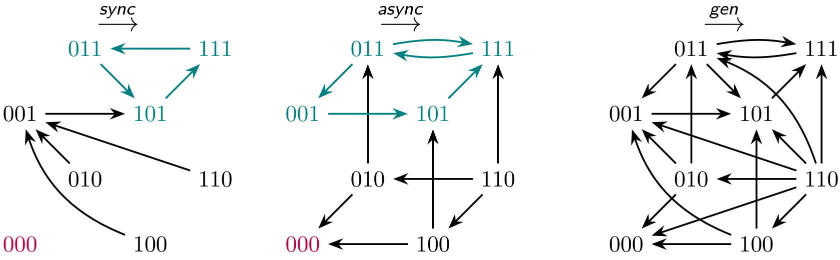

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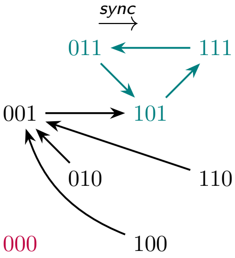

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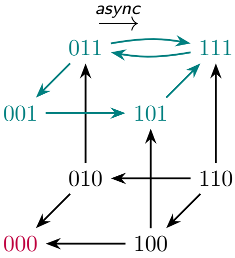

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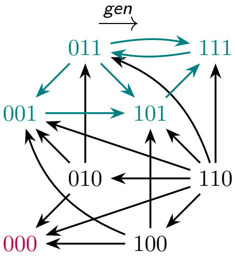

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Basins of Attraction

The **weak basin of attraction** of an attractor A is a set of configurations $\mathcal{WB}(A)$ defined as:

$$\mathcal{WB}(A) \stackrel{\Delta}{=} \{\mathbf{x} \in \mathbb{B}^n \mid \exists \mathbf{y} \in A, \mathbf{x} \rightarrow^* \mathbf{y}\}$$

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where $\mathcal{A}$ is the set of all attractors of the BN.

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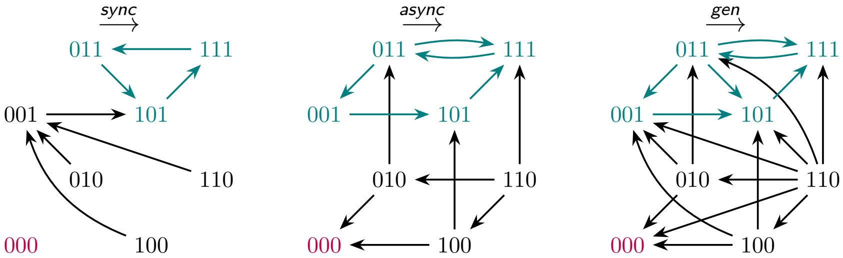

Trap Spaces

A **subspace** of $\mathbb{B}^n$ is a partial function $d\colon \{1,\dots,n\} \hookrightarrow \mathbb{B}$.
The variables on which d is defined, $D = range(d) \subseteq \{1,\dots,n\}$, are called **fixed**.
The variables which are not fixed, $F = \{1,\dots,n\} \setminus D$, are called **free**.
We denote the subspaces by vectors $\mathbf{d} \in \{0,1,*\}^n$ such that for each $i \in \{1,\dots,n\}$:

$$\mathbf{d}_i \triangleq \begin{cases} 0 & \text{if } i \in D \text{ and d(i)=0} \\ 1 & \text{if } i \in D \text{ and d(i)=1} \\ * & \text{if } i \in F \end{cases}$$

By abuse of notation, a subspace is also a set of configurations it contains, $\mathbf{d} = \{\mathbf{x} \in \mathbb{B}^n \mid \forall i \in D, \mathbf{d}_i = \mathbf{x}_i\}$.

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A **trap space** is a subspace which is also a trap set.