Interaction Graphs (Influence Graphs)

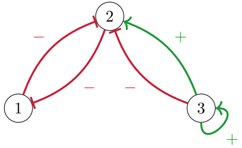
Juri Kolčák

Friday 21st November, 2025

Interaction Graphs

An **interaction graph** of dimension $n \in \mathbb{N}$ (on n vertices) is a graph $G = (V, E)$, where:

- $V = \{v_1, \ldots, v_n\}$ is the vertex set (often simply $V = \{1, \ldots, n\}$);
- $E \subseteq V \times \{+, -\} \times V$ is the set of directed labelled (signed) edges;



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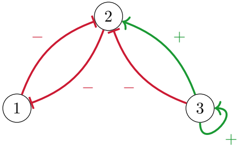

A (directed) cycle is a finite sequence of edges $C = (u_1 \xrightarrow{s_1} v_1, \dots, u_k \xrightarrow{s_k} v_k)$ such that:

- $\forall 1 \leq i < k,\ u_{i+1} = v_i$ (Directed walk);
- $u_1 = v_k$ (Closed);
- $\forall 1 \leq i \neq j \leq k,\ u_i \neq u_j$ (Simple circuit);

We write $s(e) \in \{+, -\}$ to denote the sign of an edge and $s(C) \triangleq \prod_{e \in C} s(e)$ the sign of a cycle.

Interactions

NOTATIONS:

Let $\mathbf{x}, \mathbf{y} \in \mathbb{B}^n$ be arbitrary, $\Delta(\mathbf{x}, \mathbf{y}) = \{i \in \{1, \ldots, n\} \mid \mathbf{x}_i \neq \mathbf{y}_i\}$ is the set of all variables that have different values in $\mathbf{x}$ and $\mathbf{y}$ (difference).

We us the following notation $\mathbf{x}^{\overline{W}}$ to denote the configuration which differs from $\mathbf{x}$ exactly on variables in $W \subseteq \{1, \ldots, n\}$.

For any $\mathbf{x}, \mathbf{y} \in \mathbb{B}^n$, $\mathbf{x}^{\overline{\Delta(\mathbf{x},\mathbf{y})}} = \mathbf{y}$.

In case $W = \{i\}$ is a singleton, we write simply $\mathbf{x}^{\bar{i}}$ instead of $\mathbf{x}^{\overline{\{i\}}}$.

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A variable $i \in \{1, \dots, n\}$ interacts with variable $j \in \{1, \dots, n\}$ (there is an interaction from i to j) if and only if there exists a configuration $\mathbf{x} \in \mathbb{B}^n$ such that $f_j(\mathbf{x}) \neq f_j(\mathbf{x}^{\bar{i}})$.
"The isolated action of variable i causes a change in the variable j."

The set of all variables which interact with variable i is
$\omega(\mathrm{i}) \triangleq \left\{j \in \{1, \dots, n\} \mid \exists \mathbf{x} \in \mathbb{B}^n, f_i(\mathbf{x}) \neq f_i(\mathbf{x}^{\bar{j}})\right\}$.
The set of all variables which variable i interacts with is
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Local Monotony

INTERACTION SIGNS:

An interaction from i to j is **positive**, if for all configurations $\mathbf{x} \in \mathbb{B}^n$,
$\mathbf{x}_i = 0 \Rightarrow f_j(\mathbf{x}) \leq f_j(\mathbf{x}^{\bar{i}})$. $=$ $x_i \leq x^{\bar{i}}_i \Rightarrow f_j(x) \leq f_j(x^{\bar{i}})$
"An increase of the value of the source, cannot cause decrease of the value of the target."

An interaction from i to j is **negative**, if for all configurations $\mathbf{x} \in \mathbb{B}^n$,
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Rule of a thumb:

If the local function f_j is non-decreasing in the i-th component, then $\mathbf{x}_i$ will only appear as a positive (non-negated) literal in its DNF*.
If the local function f_j is non-increasing in the i-th component, then $\mathbf{x}_i$ will only appear as a negative (negated) literal in its DNF*.

* Provided the DNF is "minimal".

Interaction Graph of a Boolean network

Given a Boolean network f of dimension n, its interaction graph $G(f) = (V, E)$ is such that:

- $V = \{1, \dots, n\}$;
- $i \xrightarrow{+} j \in E \overset{\Delta}{\iff} \exists \mathbf{x} \in \mathbb{B}^n, \mathbf{x}_i = 0$ and $f_j(\mathbf{x}) < f_j(\mathbf{x}^{\bar{i}})$;
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$$
\begin{aligned}
f_1(\mathbf{x}) &= \mathbf{x}_3 \wedge (\neg \mathbf{x}_1 \vee \neg \mathbf{x}_2) \\
f_2(\mathbf{x}) &= \mathbf{x}_1 \wedge \mathbf{x}_3 \\
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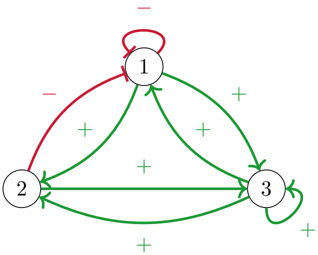

Acyclic Interaction Graphs

F. Robert. Iterations sur des ensembles finis et automates cellulaires contractants.
Linear Algebra and its Applications, 29:393–412, 1980

Let f be a Boolean network of dimension n and $G(f) = (V, E)$ its influence graph. If $G(f)$ is acyclic, then every configuration converges to the same attractor, a fixed point.

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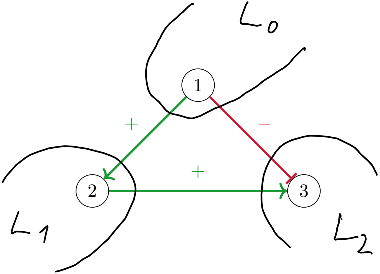
EXAMPLE:

$$f_1(\mathbf{x}) = 1$$
$$f_2(\mathbf{x}) = \mathbf{x}_1$$
$$f_3(\mathbf{x}) = \neg\mathbf{x}_1 \vee \mathbf{x}_2$$



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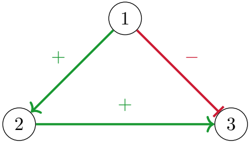


$$000 \xrightarrow{sync} 101 \xrightarrow{sync} 110 \xrightarrow{sync} 111$$

Multi-stationarity and Positive Cycles

"For a Boolean network to have multiple fixed points, its interaction graph has to have a positive cycle." – René Thomas, 1980 (conjecture)

Synchronous semantics:

J. Aracena, J. Demongeot, and E. Goles. Positive and negative circuits in discrete neural networks.
IEEE Transactions on Neural Networks, 15(1):77–83, 2004

J. Aracena. Maximum number of fixed points in regulatory boolean networks.
Bulletin of Mathematical Biology, 70(5):1398–1409, Jul 2008

Fully asynchronous semantics:

Élisabeth Remy, P. Ruet, and D. Thieffry. Graphic requirements for multistability and attractive cycles in a boolean dynamical framework.
Advances in Applied Mathematics, 41(3):335–350, 2008

A. Richard and J.-P. Comet. Necessary conditions for multistationarity in discrete dynamical systems.
Discrete Applied Mathematics, 155(18):2403–2413, 2007

A. Richard. Positive circuits and maximal number of fixed points in discrete dynamical systems.
Discrete Applied Mathematics, 157(15):3281–3288, 2009

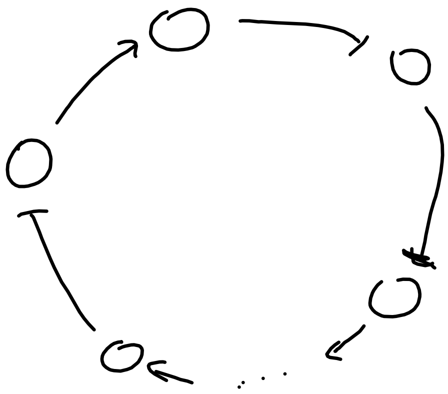
Generalised asynchronous semantics:

M. Noual. *Updating Automata Networks.*
PhD thesis, Ecole normale supérieure de Lyon - ENS LYON, Jun 2012

Assume $G(f)$ does not have any positive cycle

↳ $G(f)$ has no cycle ⇒ Robert 1980

↳ $G(f)$ has a negative cycle:



$s(C) = -1$

$\forall i \in \{1, \dots, n\}$

$f_i(x) = x_i$

$x \in \mathbb{B}^n$ is a fixed point ⇔ $f(x) = x$

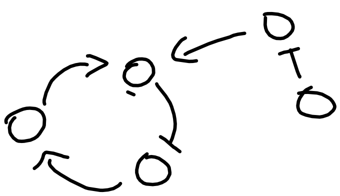
Whenever we see $i \xrightarrow{+} j$ ⇒ j wants to be equal to i

Whenever we see $i \overset{-}{\dashv} j$ ⇒ j wants to differ from i

Along a negative cycle, there is an odd number of "flips" = negative interactions

so we end with opposite value for our "starting point"

⇒ "There is always at least frustrated varable in a negative cycle."



⇒ "lock" the cycle into a fixed point

We can iterate the above arguments for any subgraphs "feeding" into the negative cycle.

Bounds on the Number of Attractors

J. Aracena. Maximum number of fixed points in regulatory boolean networks.
Bulletin of Mathematical Biology, 70(5):1398–1409, Jul 2008

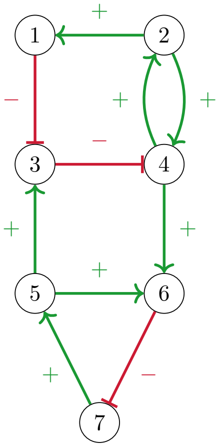
A. Richard. Positive circuits and maximal number of fixed points in discrete dynamical systems.
Discrete Applied Mathematics, 157(15):3281–3288, 2009

J. Aracena, A. Richard, and L. Salinas. Number of fixed points and disjoint cycles in monotone boolean networks.
SIAM Journal on Discrete Mathematics, 31(3):1702–1725, 2017

Given an interaction graph G, the following bounds hold for the **maximum number of fixed points** of any BN with G as its interaction graph ($\mathtt{maxFP}(G)$):

- $\mathtt{maxFP}(G)$ is lower bounded by the **packing number** of G, $\nu(G)$, i.e. the maximum number of disjoint cycles in G, as follows: $\nu(G) + 1 \leq \mathtt{maxFP}(G)$;
- $\mathtt{maxFP}(G)$ is upper bounded by the **positive feedback vertex set** of G, $\tau^+(G)$, i.e. the minimum number of vertices needed to intersect every positive cycle in G, as follows: $\mathtt{maxFP}(G) \leq 2^{\tau^+(G)}$; This upper bound applies to **all attractors** (including limit cycles) under the **fully asynchronous semantics**.

Bounds on the Number of Attractors – Example



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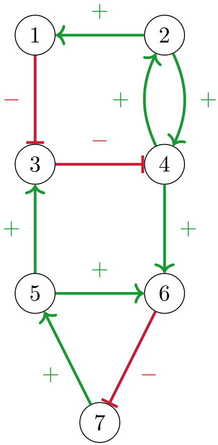

$3 \leq \texttt{maxFP}(G)$

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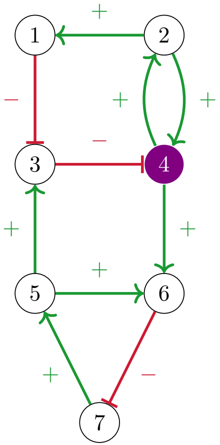

$$3 \leq \texttt{maxFP}(G) \leq 2^1 = 2$$

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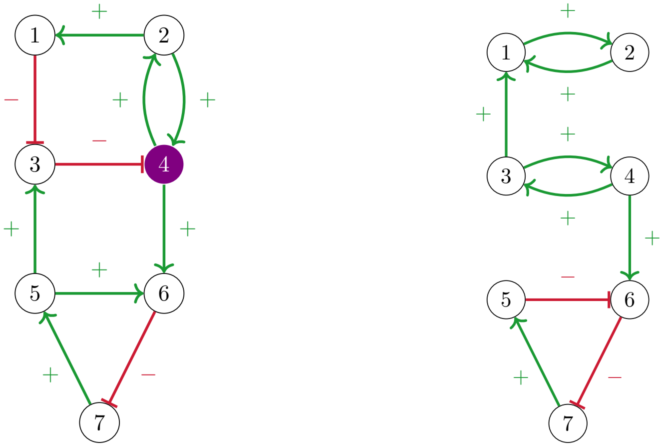

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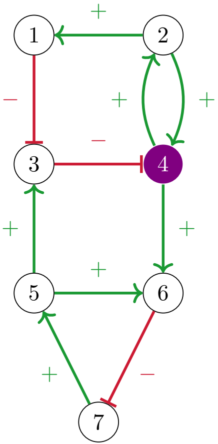
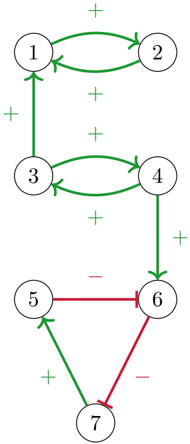
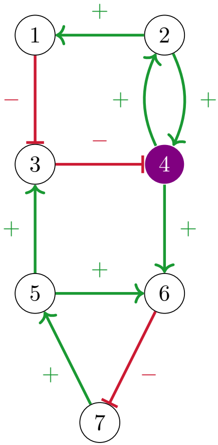




$3 \leq \texttt{maxFP}(G) \leq 2^1 = 2$

$4 \leq \texttt{maxFP}(G')$

Bounds on the Number of Attractors – Example



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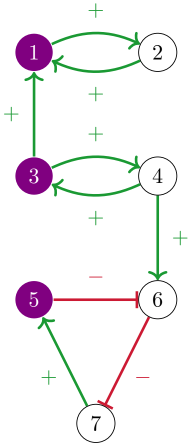


$$4 \leq \texttt{maxFP}(G') \leq 2^3 = 8$$

Cyclic Attractors and Negative Cycles

"For a Boolean network to have a cyclic attractor (sustained oscillation), its interaction graph has to have a negative cycle." – René Thomas, 1980 (conjecture)

Fully asynchronous semantics:

A. Richard. Negative circuits and sustained oscillations in asynchronous automata networks.
Advances in Applied Mathematics, 44(4):378–392, 2010

Generalised asynchronous semantics:

M. Noual. *Updating Automata Networks.*
PhD thesis, Ecole normale supérieure de Lyon - ENS LYON, Jun 2012

S. Sené. *Sur la bio-informatique des réseaux d'automates.*
Accreditation to supervise research, Université d'Evry-Val d'Essonne, Nov 2012

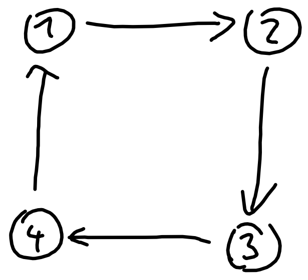
Normal Transitions

M. Noual and S. Sené. Synchronism versus asynchronism in monotonic boolean automata networks. *Natural Computing*, 17(2):393–402, Jun 2018

↱ Changes at least 2 variables

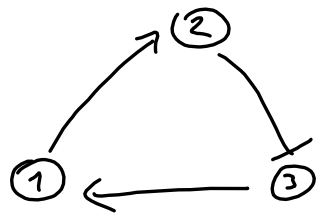
A synchronous (generalised) transition $\mathbf{x} \to \mathbf{y}$ is **normal**, if $\mathbf{y}$ is not reachable from $\mathbf{x}$ in fully asynchronous semantics.

Existence of normal transition is tied to existence of NOPE-cycles (negative-odd or positive-even) in the interaction graph $G(f)$.



0101

all variables are frustrated



011

A normal transition $x \to y$ induces a NOPE-cycle

$H = (\Delta(x,y),\ \mathrm{frus}(x) \cap \Delta(x,y) \times \{+,-\} \times \Delta(x,y))$

"frustrated" interactions — restrict edges to the ones between variables in $\Delta(x,y)$

$i \to j$ is frustrated in x if:

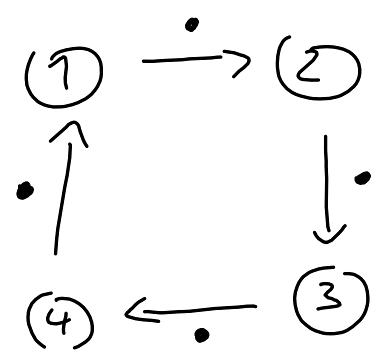
$x_i = x_j$ and $s(i \to j) = -1$

$x_i \neq x_j$ and $s(i \to j) = +1$

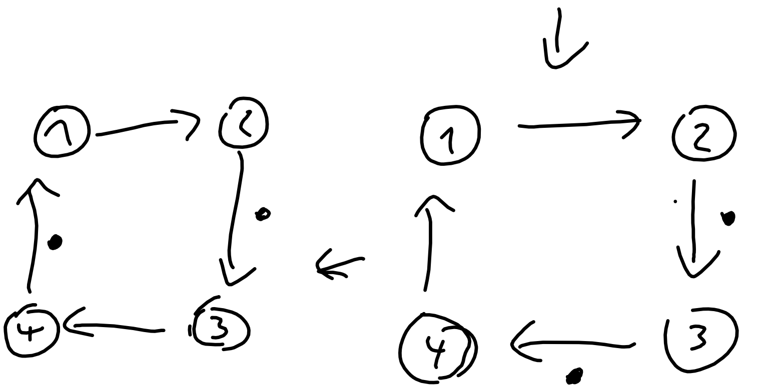
If H is acyclic, we can layer it and use this partial order to reproduce $x \to y$ by a sequence of fully asynchronous transitions.

-> We start updating from the variables that have no outgoing interactions in H, so that we do not resolve frustrations "prematurely"

A cycle in H is necessarily a NOPE cycle



0101 $\xrightarrow{\text{normal}}$ 1010



1101

1100

Whenever a variable is updated the outgoing and incoming interaction frustrations flip

(This only holds for cycles)

=> Asynchronous updates can only "shift" a frustration along a cycle, or remove 2 frustrations

Crucially, asynchronous updates cannot introduce more frustrated interaction

Normal Transitions

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Impact of normal transitions

1. **No impact**: No changes to attractors or their basins;
2. **Freedom-impact**: The basins of attraction of attractors reachable from $\mathbf{y}$ grow;
3. **Destruction-impact**: A cyclic attractor containing $\mathbf{x}$ is "emptied" into other attractor(s);
4. **Growth-impact**: A cyclic attractor containing $\mathbf{x}$ grows, absorbing more configurations, including $\mathbf{y}$;