Boolean Networks in Life Sciences
2025–2026 Compulsory Exercise Sheet

Each exercise is worth 1 point (12.5%).
Partial answers will receive fractional points.
At least 50% is required for successful completion.

Exercise 1 *Consider the following statements in Boolean algebra and determine whether they are valid (whether the equality holds). Justify your reasoning (proof in case the statement is true, counterexample in case it is false).*

1. $(a \lor (b \land c)) \land (a \lor (b \land \neg c)) \stackrel{?}{=} a$
2. $a \land (b \lor c) \stackrel{?}{=} (a \land b \land \neg c) \lor (a \land \neg b \land c)$

Exercise 2 *Consider the Boolean network f of dimension 3 given by the following local functions:*

$$
\begin{aligned}
f_1(\mathbf{x}) &= \neg \mathbf{x}_1 \land \neg \mathbf{x}_3 \\
f_2(\mathbf{x}) &= \neg \mathbf{x}_1 \land \mathbf{x}_2 \\
f_3(\mathbf{x}) &= \mathbf{x}_1 \lor (\mathbf{x}_2 \land \mathbf{x}_3)
\end{aligned}
$$

Construct a transition system of f for each of the following semantics: synchronous, fully asynchronous, generalised asynchronous and most permissive.

Determine the attractors of f under each of the four semantics.

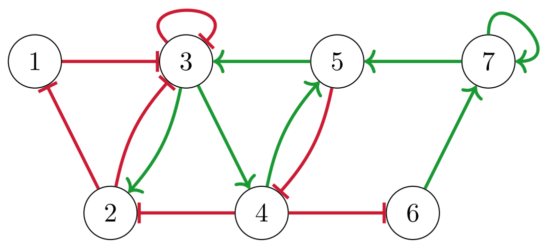
Exercise 3 *Consider the Boolean network g of dimension 3 given by the following local functions:*

$$
\begin{aligned}
g_1(\mathbf{x}) &= \neg \mathbf{x}_1 \lor \neg \mathbf{x}_3 \\
g_2(\mathbf{x}) &= \neg \mathbf{x}_1 \lor \neg \mathbf{x}_2 \\
g_3(\mathbf{x}) &= \neg \mathbf{x}_1 \lor \neg \mathbf{x}_2
\end{aligned}
$$

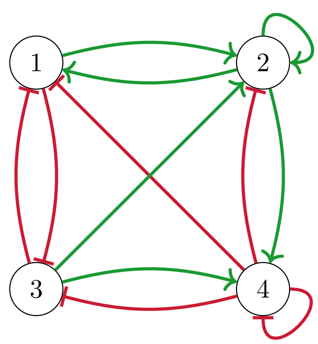
Construct the interaction graph $G(g)$ of g.

With the help of the interaction graph $G(g)$, identify all normal transitions of g and classify them by their impact (none, destruction, freedom or growth).

Exercise 4 *Consider the following interaction graph G and use the packing number $\nu(G)$ (the maximum number of disjoint cycles) and the positive feedback vertex set $\tau^+(G)$ (the minimum number of vertices needed to intersect every positive cycle) to determine the lower and upper bounds of the maximal number of fixed points a Boolean network f' with $G = G(f')$ as its interaction graph may have.*



Exercise 5 *Consider the following interaction graph H and determine the extreme parameters $p^i_{\omega^+(i)}$ and $p^i_{\omega^-(i)}$ for each variable $i \in \{1, 2, 3, 4\}$.*



Compute additionally the abstract parameter set (restricted to variable 1) monotonicity narrowing, $\Lambda_m(\langle \mathbf{l}, \mathbf{u} \rangle)$ on the following $\mathbf{l}$ and $\mathbf{u}$, corresponding to having taken the transition $1101 \longrightarrow 0101$:

	$p^1_\emptyset$	$p^1_{\{2\}}$	$p^1_{\{3\}}$	$p^1_{\{2,3\}}$	$p^1_{\{4\}}$	$p^1_{\{2,4\}}$	$p^1_{\{3,4\}}$	$p^1_{\{2,3,4\}}$
$\mathbf{u}$	1	1	1	1	1	0	0	1
$\mathbf{l}$	0	1	0	0	0	0	0	0

Exercise 6 *Consider the Boolean network h of dimension 3 given by the following local functions with **asynchronous** semantics:*

$$h_1(\mathbf{x}) = \mathbf{x}_2$$
$$h_2(\mathbf{x}) = \neg\mathbf{x}_1 \wedge \neg\mathbf{x}_3$$
$$h_3(\mathbf{x}) = \neg\mathbf{x}_1$$

Illustrate the execution tree (a "complete prefix" of) rooted in the configuration $\mathbf{x} = 000$.

Exercise 7 *Consider the Boolean network h from Exercise 6, again with asynchronous semantics, and the following LTL and CTL formulae:*

1. $\mathbf{F}(\mathbf{x}_2 \wedge (\mathbf{x}_2 \,\mathbf{U}\, \mathbf{x}_3))$
2. $\mathbf{G}((\mathbf{x}_2 \wedge \mathbf{X}(\neg\mathbf{x}_2)) \implies \mathbf{X}(\neg\mathbf{x}_2 \,\mathbf{U}\, \neg\mathbf{x}_1))$
3. $\forall\mathbf{G}(\mathbf{x}_1 \implies \exists(\mathbf{x}_1 \,\mathbf{U}\, (\neg\mathbf{x}_1 \wedge \neg\mathbf{x}_2 \wedge \neg\mathbf{x}_3)))$
4. $\forall\mathbf{F}(\exists\mathbf{G}(\mathbf{x}_3))$

Determine the validity of each of the formulae in the configuration $\mathbf{x} = 000$ of the Boolean network h. In case the formula is false (not valid), provide a counterexample.

(Note: Do not consider any fairness assumptions.)

Exercise 8 *The LTL formula number two (2.) from Exercise 7 defines a regular safety property L_{safe}.*

Construct a nondeterministic finite automaton $\mathcal{A}$ which recognises the minimal bad prefixes of L_{safe}, $L(\mathcal{A}) = \mathtt{MinBad}(L_{safe})$.